- **First-Order Linear Differential Equations**: $\frac{dy}{dx}+P(x)y=Q(x)$
  - $\mu(x)=e^{\int P(x)dx}$ $\left(\mu'(x)=P(x)e^{\int P(x)dx}\right)$ Integrating factor.
  - $\frac{dy}{dx}e^{\int P(x)dx}+P(x)e^{\int P(x)dx}y=Q(x)e^{\int P(x)dx}$ Multiply through.
  - $\frac{dy}{dx}\mu(x)+\mu'(x)y=Q(x)\mu(x)$ Rewrite
  - $(\mu(x)y)'=Q(x)\mu(x)$ Use product rule!
  - $y=\frac{1}{\mu(x)}\int Q(x)\mu(x)dx$ Solution
  - Further idea:
    - $y=e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$ Add *C*.
    - $y=e^{-\int P(x)dx}\left(\int Q(x)e^{\int P(x)dx}dx+C\right)$
    - $y=e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx+Ce^{-\int P(x)dx}$
    - Note that this solution contains two parts.
    - One part is the **steady-state (long term) solution** $e^{-\int P(x)dx}\int Q(x)e^{\int P(x)dx}dx$, which predicts long term behavior and does not depend on the initial condition.
    - The other part is the **transient** $Ce^{-\int P(x)dx}$, which depends on the initial condition, but will approach zero in the long term. Note, *C* corresponds to the initial condition.
  - Applications:
    - Fluid friction and air resistance
    - Electrical circuits
    - Capacitor and inductor problems (RC circuit, RL circuit)
    - Diffusion
- **Bernoulli Differential Equations**
  - $\frac{dy}{dx}+P(x)y=Q(x)y^n$
  - Substitution: $z=y^{1-n}$ $dz=(1-n)y^{-n}dy$
  - Then solve as a first-order linear differential equation
  - Applications:
    - Fluid dynamics
    - Leaking tank (using Bernoulli's principle)